ON THE STABILITY OF MOTION OF A HORIZONTAL GYROCOMPASS IN THE PRESENCE OF DISSIPATIVE FORCES

(OB USTOICHIVOSTI DVIZHENIA GIROGORIZONTKOMPASA PRI NALICHII DISSIPATIVNYKH SIL)

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This paper gives the investigation of the asymptotic stability of the unperturbed motion of a horizontal gyrocompass when small dissipative forces are taken into account.

There are considered cases when the base of a gyrocompass is fixed with respect to the earth, speeds are constant and circulatory motions are uniform.

1. When the damping of the natural vibrations is neglected then the equations of motion of a horizontal gyrocompass of the Geckeler-Anschütz type which have been given in [1,2] are of the following form

$$\frac{Pl}{g}\frac{d}{dt}(V\alpha) - Pl\beta - \Omega 2B\sin\varepsilon^\circ\delta = 0, \qquad \dot\beta + \frac{V}{R}\alpha - \Omega\gamma = 0$$
$$\dot\gamma + \frac{2B\sin\varepsilon^\circ g}{PlR}\delta + \Omega\beta = 0, \qquad \frac{d}{dt}(2B\sin\varepsilon^\circ\delta) - Pl\gamma + \Omega\frac{Pl}{g}V\alpha = 0 \qquad (1.1)$$

The parameter ε° satisfies the condition

$$2B\cos\varepsilon^\circ = \frac{Pl}{g}V \qquad (1.2)$$

This condition is autonomously satisfied for gyroscopes of the type which we consider. The notation used in (1.1) is the same as in [1,2,3].

Let us introduce for convenience the new variables α_1 and δ_1 through the relations

$$V\alpha = \alpha_1, \qquad \frac{2B\sin\varepsilon^\circ g}{Pl}\delta = \delta_1 \qquad (1.3)$$

Substituting (1.3) in (1.1) we obtain

$$\begin{aligned} &\dot{\alpha}_1 - g\beta - \Omega\delta_1 = 0, \qquad \dot{\beta} + \frac{\alpha_1}{R} - \Omega\gamma = 0 \\ &\dot{\gamma} + \frac{\delta_1}{R} + \Omega\beta = 0, \qquad \dot{\delta}_1 - g\gamma + \Omega\alpha_1 = 0 \end{aligned} \tag{1.4}$$

In the equation (1.4) the coefficients of the unknown functions can be regarded as constants, with the exception of Ω which is determined by

$$\Omega = U \sin\varphi + \frac{v_E}{R} \operatorname{tg}\varphi + \frac{d\alpha^*}{dt} \qquad \left(\alpha^* = \frac{v_N}{RU\cos\varphi + v_E}\right) \tag{1.5}$$

In general, when a ship moves on the earth's surface, Ω is a continuous function of time t. The system (1.4) is equivalent to two second order equations

$$\begin{aligned} &\ddot{\alpha}_1 + \lambda\alpha_1 - 2\Omega\dot{\delta}_1 - \dot{\Omega}\delta_1 = 0 \\ &\ddot{\delta}_1 + \lambda\delta_1 + 2\Omega\dot{\alpha}_1 + \dot{\Omega}\alpha_1 = 0 \end{aligned} \qquad (\lambda = \nu^2 - \Omega^2) \tag{1.6}$$

Simultaneously with the system (1.6) we shall analyze also the system

$$\begin{aligned} &\ddot{\alpha}_1 + 2b_1\dot{\alpha}_1 + \lambda\alpha_1 - 2\Omega\dot{\delta}_1 - \dot{\Omega}\delta_1 = 0 \\ &\ddot{\delta}_1 + 2b_2\dot{\delta}_1 + \lambda\delta_1 + 2\Omega\dot{\alpha}_1 + \dot{\Omega}\alpha_1 = 0 \end{aligned} \tag{1.7}$$

where the coefficients b_1 and b_2 reflect the action of dissipative forces. We shall assume that these coefficients are arbitrarily small positive numbers.

2. We shall consider first a simple case when Ω is a constant. Under this assumption the system (1.7) becomes

$$\ddot{\alpha}_1 + 2b_1\dot{\alpha}_1 + \lambda\alpha_1 - 2\Omega\dot{\delta}_1 = 0, \qquad \ddot{\delta}_1 + 2b_2\dot{\delta}_1 + \lambda\delta_1 + 2\Omega\dot{\alpha}_1 = 0 \tag{2.1}$$

After replacing λ by its values as obtained from (1.6) the characteristic equation of (2.1) is

$$\begin{aligned} &D^4 + 2(b_1 + b_2) D^3 + 2(2b_1b_2 + \nu^2 + \Omega^2) D^2 + \\ &+ 2(b_1 + b_2)(\nu^2 - \Omega^2) D + (\nu^2 - \Omega^2)^2 = 0 \end{aligned} \tag{2.2}$$

Applying the Hurwitz criterion it can be easily shown that if b_1 and b_2 are positive constants and if the condition (obtained by different methods in [4,5])

$$\Omega < \nu \tag{2.3}$$

The trivial solution of the system (2.2) will be then asymptotically stable. When

$$\Omega > \nu \tag{2.4}$$

(the case $\Omega = \nu$ which is on the boundary of stability and instability will not be considered), then the system will be unstable even with smallest dissipative forces present. Indeed the system

$$\ddot{\alpha}_1 + \lambda\alpha_1 - 2\Omega\dot{\delta}_1 = 0, \qquad \ddot{\delta}_1 + \lambda\delta_1 + 2\Omega\dot{\alpha}_1 = 0 \tag{2.5}$$

obtained from (2.1) on the assumption of the absence of damping is a special case of a system where the conservative and gyroscopic forces are [9]

$$\ddot{x}_j = -\lambda_j x_j + \sum_k g_{jk} \dot{x}_k \qquad (g_{jk} = -g_{kj}) \tag{2.6}$$

When $\Omega > \nu$, then by (1.6) we have $\lambda < 0$, and the system (2.5) acted only by conservative forces is unstable. The degree of instability (that is the number of negative λ's) is even and equals two.

When the degree of instability is even and fully dissipative forces are absent, then the system can be stabilized by a suitable choice of gyroscopic forces. In our case these forces are expressed by the terms $2\Omega\dot{\delta}_1$ and $-2\Omega\dot{\alpha}_1$. This stabilization, however, is only (Kelvin's terminology) "temporary" and it is destroyed in the presence of the smallest dissipative forces. For a horizontal gyrocompass on a base which is fixed with respect to the earth we have $\Omega \equiv U \sin\varphi$ = const, $V \equiv RU \cos\varphi$ = const, hence the condition (2.3) reduces to

$$U \sin\varphi < \nu \tag{2.7}$$

The above condition assures the stability of a horizontal gyrocompass when dissipative forces are present. Since

$$\nu = \sqrt{g/R} \approx 1.24 \cdot 10^{-3}\,\mathrm{sec}^{-1}, \qquad U \approx 7.29 \cdot 10^{-5}\,\mathrm{sec}^{-1}$$

the inequality (2.7) is always satisfied.

When a moving ship has a constant eastern velocity component v_E, then the condition (2.3) becomes

$$U \sin\varphi\,(1 + v_E/RU \cos\varphi) < \nu \tag{2.8}$$

This condition is also always satisfied in practical cases.

3. We shall investigate now the case when V and Ω vary and are function of the time t. We shall assume that the coefficients of dissipation b_1 and b_2 are so small compared with ν, that their distinctness can be neglected and we can set $b_1 = b_2 = b$.

Then from (1.7) we obtain the system of equations

$$\begin{aligned} &\ddot{\alpha}_1 + 2b\dot{\alpha}_1 + (\nu^2 - \Omega^2)\alpha_1 - 2\Omega\dot{\delta}_1 - \dot{\Omega}\delta_1 = 0 \\ &\ddot{\delta}_1 + 2b\dot{\delta}_1 + (\nu^2 - \Omega^2)\delta_1 + 2\Omega\dot{\alpha}_1 + \dot{\Omega}\alpha_1 = 0 \end{aligned} \tag{3.1}$$

We shall introduce in (3.1) the new variables ξ_1 and ξ_2 by the nonsingular transformation [2]

$$\xi_1 = \alpha_1\cos\theta - \delta_1\sin\theta, \qquad \xi_2 = \alpha_1\sin\theta + \delta_1\cos\theta \qquad \left(\theta = \int_0^t \Omega(\tau)\,d\tau\right) \tag{3.2}$$

Hence

$$\alpha_1 = \xi_1\cos\theta + \xi_2\sin\theta, \qquad \delta_1 = -\xi_1\sin\theta + \xi_2\cos\theta \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1) we obtain the system of equations in ξ_1 and ξ_2

$$\ddot{\xi}_1 + 2b\dot{\xi}_1 + \nu^2\xi_1 + 2b\Omega(t)\xi_2 = 0, \qquad \ddot{\xi}_2 + 2b\dot{\xi}_2 + \nu^2\xi_2 - 2b\Omega(t)\xi_1 = 0 \tag{3.4}$$

Multiplying the second equation in (3.4) by $-i$ $(i = \sqrt{(-1)})$ and adding to the first equation we obtain

$$\ddot{w} + 2b\dot{w} + (\nu^2 + 2b\Omega(t)i)w = 0 \qquad (w = \xi_1 - i\xi_2) \tag{3.5}$$

4. We shall consider now the case of consecutive left circulations of a ship moving with constant speed v, whose course was initially ψ_0.

The northern and eastern velocity components of our ship are, respectively

$$v_N = v\cos(\psi_0 - \omega t), \qquad v_E = v\sin(\psi_0 - \omega t) \qquad \left(\omega = \frac{2\pi}{T}\right) \tag{4.1}$$

where ω is the angular velocity of circulation, T is the period of circulation. Under these conditions ν and Ω are periodic functions of period T.

For example, if a circulation begins from the southern course we substitute in (4.1) $\psi_0 = 180^\circ$, hence $v_N = v\cos\omega t$, $v_E = v\sin\omega t$.

It has been shown in [3] that for a ship moving in circles we can simplify (1.5) by retaining only the principal part of the term representing the rate of change of the velocity direction.

For a circulation beginning from the southern course we would have

$$\Omega = \mu\omega \sin \omega t \qquad (\mu = v \,/\, RU \cos\varphi) \tag{4.2}$$

Further, in accordance with (3.2) we obtain

$$\theta = \mu\omega \int_0^t \sin \omega\tau \, d\tau = \mu\,(1 - \cos \omega t) \tag{4.3}$$

Investigating the stability of a horizontal gyrocompass moving in circles, we begin by using the method of mean values and taking the value of Ω as given by (4.2). Substituting for Ω its mean value over a period of one circulation, which by (4.2) equals zero we obtain the system of mean values equations

$$\ddot{u}_1 + 2b\dot{u}_1 + \nu^2 u_1 = 0, \qquad \ddot{u}_2 + 2b\dot{u}_2 + \nu^2 u_2 = 0 \tag{4.4}$$

which corresponds to the system (3.4).

When $b < \nu$, then the equations (4.4) have the following solutions

$$\begin{aligned} u_1 &= e^{-bt}\,(C_1 \cos qt + C_2 \sin qt) \\ u_2 &= e^{-bt}\,(C_3 \cos qt + C_4 \sin qt) \end{aligned} \qquad (q = \sqrt{\nu^2 - b^2}, \quad C_j = \mathrm{const}) \tag{4.5}$$

The roots of the characteristic equation of the system (4.4) have negative real parts when $b > 0$, therefore the use of the mean values equations is justified [7].

When a ship moves in circles and the damping is weak then the characteristic exponents of the system (3.1) can be expressed by

$$\varkappa_{1,2} = -b \pm \nu i, \qquad \varkappa_{3,4} = -b \pm \nu i \tag{4.6}$$

which are derived from (4.4) and which coincide with the characteristic exponents derived in [2] for the case when $b = 0$.

Thus, for circulatory motion, the solutions (3.4) are asymptotically stable even with smallest dissipations present.

If instead of ξ_1 and ξ_2 we use, respectively, the solutions of (4.5) which were obtained by the method of mean values, then by (3.3) we have

$$\begin{aligned}\alpha_1 &= e^{-bt}(C_1\cos qt + C_2\sin qt)\cos\theta + e^{-bt}(C_3\cos qt + C_4\sin qt)\sin\theta\\ \delta_1 &= -e^{-bt}(C_1\cos qt + C_2\sin qt)\sin\theta + e^{-bt}(C_3\cos qt + C_4\sin qt)\cos\theta\end{aligned} \tag{4.7}$$

If the damping is very weak then we can set $q = \nu$ and obtain

$$\begin{aligned}\alpha_1 &= e^{-bt}(C_1\cos\nu t + C_2\sin\nu t)\cos\theta + e^{-bt}(C_3\cos\nu t + C_4\sin\nu t)\sin\theta\\ \delta_1 &= -e^{-bt}(C_1\cos\nu t + C_2\sin\nu t)\sin\theta + e^{-bt}(C_3\cos\nu t + C_4\sin\nu t)\cos\theta\end{aligned} \tag{4.8}$$

The equations of motion of a horizontal gyrocompass in its initial form (1.1) are also asymptotically stable because the introduction of the new variables through (1.3) and (3.2) is performed by nonsingular linear transformations with periodic coefficients.

5. We shall consider now another method of investigating the stability of a horizontal gyrocompass, which is different than the mean values method. Substituting

$$w = e^{-bt}y \tag{5.1}$$

in (3.5) we obtain

$$\ddot{y} + (\nu^2 - b^2 + 2ib\Omega(t))\,y = 0 \tag{5.2}$$

Let the ship move uniformly in circles beginning from the southern course. Using (4.2) we get

$$\ddot{y} + (\nu^2 - b^2 + 2i\mu b\omega\sin\omega t)\,y = 0 \tag{5.3}$$

Setting $\omega t = 2iz$ we obtain the modified Mathieu equation with an imaginary argument

$$\frac{d^2y}{dz^2} - (p^2 - 2k^2\,\mathrm{sh}\,2z)\,y = 0 \qquad \left(p^2 = \frac{4(\nu^2 - b^2)}{\omega^2}, \quad k^2 = 4\mu\frac{b}{\omega}\right) \tag{5.4}$$

Introducing in (5.4) the new argument $\zeta = ike^z$ we obtain

$$\frac{d^2y}{d\zeta^2} + \frac{1}{\zeta}\frac{dy}{d\zeta} + \left(1 - \frac{p^2}{\zeta^2} - \frac{k^4}{\zeta^4}\right)y = 0 \tag{5.5}$$

When p is not an integer and when $k \to 0$, then we have the asymptotic solution [8] in the form

$$y = C_1'J_p(ke^z) + C_2'J_{-p}(ke^z) \qquad (C_1',\ C_2' = \mathrm{const}) \tag{5.6}$$

From (5.6) and by (5.1) we obtain

$$w = e^{-bt}(C_1' J_p(ke^z) + C_2' J_{-p}(ke^z)) \tag{5.7}$$

Assuming that the damping is weak we can set on the strength of (5.4) $p \approx 2\nu/\omega$.

Using further the asymptotic formulas for the functions $J_p(x)$ and $J_{-p}(x)$ [6] for the small values of the argument

$$J_p(x) \underset{x\to 0}{\sim} \frac{x^p}{2^p \Gamma(1+p)}, \qquad J_{-p}(x) \underset{x\to 0}{\sim} \frac{x^{-p}}{2^{-p}\Gamma(1-p)} \tag{5.8}$$

we obtain

$$\begin{aligned} J_p(ke^z) &\sim \left(\frac{k}{2}\right)^p \frac{1}{\Gamma(1+p)}(\cos \nu t - i \sin \nu t) \\ J_{-p}(ke^z) &\sim \left(\frac{k}{2}\right)^{-p} \frac{1}{\Gamma(1-p)}(\cos \nu t + i \sin \nu t) \end{aligned} \tag{5.9}$$

Hence by (5.7) we shall have

$$w = e^{-bt}(D_1 \cos \nu t + i D_2 \sin \nu t) \tag{5.10}$$

where

$$\begin{aligned} D_1 &= C_1' \left(\frac{k}{2}\right)^p \frac{1}{\Gamma(1+p)} + C_2' \left(\frac{k}{2}\right)^{-p} \frac{1}{\Gamma(1-p)} \\ D_2 &= C_2' \left(\frac{k}{2}\right)^{-p} \frac{1}{\Gamma(1-p)} - C_1' \left(\frac{k}{2}\right)^p \frac{1}{\Gamma(1+p)} \end{aligned} \tag{5.11}$$

Setting

$$D_1 = K_1 - iK_3, \qquad D_2 = -(K_4 + iK_2) \tag{5.12}$$

and using (3.5) we have

$$\xi_1 = e^{-bt}(K_1 \cos \nu t + K_2 \sin \nu t), \qquad \xi_2 = e^{-bt}(K_3 \cos \nu t + K_4 \sin \nu t) \tag{5.13}$$

By the inverse transformation formula (3.3) we obtain finally

$$\begin{aligned} \alpha_1 &= e^{-bt}(K_1 \cos \nu t + K_2 \sin \nu t)\cos\theta + e^{-bt}(K_3 \cos \nu t + K_4 \sin \nu t)\sin\theta \\ \delta_1 &= -e^{-bt}(K_1 \cos \nu t + K_2 \sin \nu t)\sin\theta + e^{-bt}(K_3 \cos \nu t + K_4 \sin \nu t)\cos\theta \end{aligned} \tag{5.14}$$

where the constants K_j are determined from the initial conditions. These expressions coincide with the previous results (4.7) obtained by a different method.

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